Imp.

EQUATION OF TANGENT (CALCULUS METHOD).

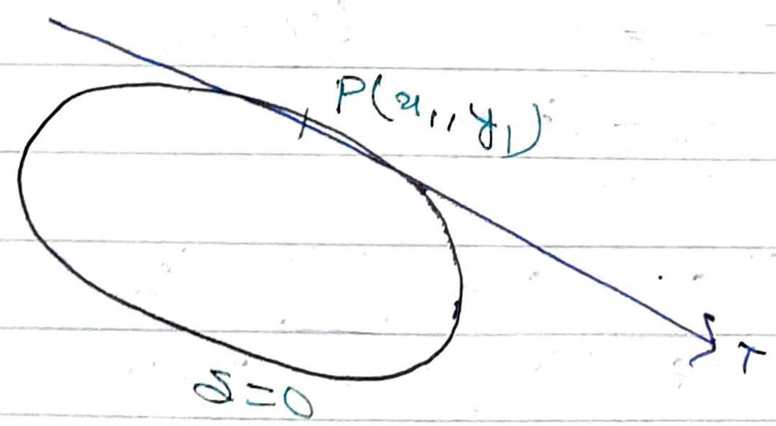
THEOREM:

To find the equation of a tangent to the conic

$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

at the pt. (x_1, y_1)

Proof:



Given equation of curve is

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. —— (1)

diff (1) partially w.r.t. x and y. we get

$$\frac{\partial S}{\partial x} = 2ax + 2h\frac{\partial (xy)}{\partial x} + b\frac{\partial}{\partial x}(y^2) + 2g\frac{\partial}{\partial x}(x) + 2f\frac{\partial y}{\partial x} + \frac{\partial (c)}{\partial x}$$

$$= 2ax + 2hy\frac{\partial x}{\partial x} + b\cdot 0 + 2\cdot g\cdot 1 + 2f\cdot 0 + 0$$

$$= 2ax + 2hy + 2g$$

$$= 2(ax + hy + g)$$

$$\frac{\partial S}{\partial y} = a\frac{\partial}{\partial y}(x^2) + 2h\frac{\partial}{\partial y}(xy) + b\frac{\partial}{\partial y}(y^2) + 2g\frac{\partial}{\partial y}(x) + 2f\frac{\partial (y)}{\partial y} + \frac{\partial}{\partial y}(c)$$

$$= 2hx + 2by + 2f.$$

$$= 2(hx + by + f)$$

Let PT be the tangent to the conic at any point $P(x_1, y_1)$ then its eqn is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

$$= -\left(\frac{\frac{\partial S}{\partial x}}{\frac{\partial S}{\partial y}}\right)_{(x_1, y_1)} (x - x_1)$$

$$= -\left\{\frac{(ax + hy + g)}{(hx + by + f)}\right\}_{(x_1, y_1)} (x - x_1)$$

$$= -\left(\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}\right)\cdot(x - x_1)$$

or, $(y - y_1)(hx_1 + by_1 + f) = -(x - x_1)(ax_1 + hy_1 + g)$

or, $hyx_1 + byy_1 + fy - hx_1y_1 - by_1^2 - fy_1$
$= -axx_1 - hxy_1 - gx + ax_1^2 + hx_1y_1 + gx_1$

or, $axx_1 + hx_1y + hxy_1 + byy_1 + gx + fy$
$= ax_1^2 + 2hx_1y_1 + by_1^2 + gx_1 + fy_1$

or, ~~$axx_1 +$~~
Adding $gx_1 + fy_1 + c$ both sides, we get

$axx_1 + hx_1y + hxy_1 + byy_1 + gx + gx_1 + fy_1 + c + fy$
$= ax_1^2 + 2hx_1y_1 + by_1^2 + gx_1 + gx_1 + fy_1$
$+ fy_1 + c.$

or, $axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1)$
$+ c = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$

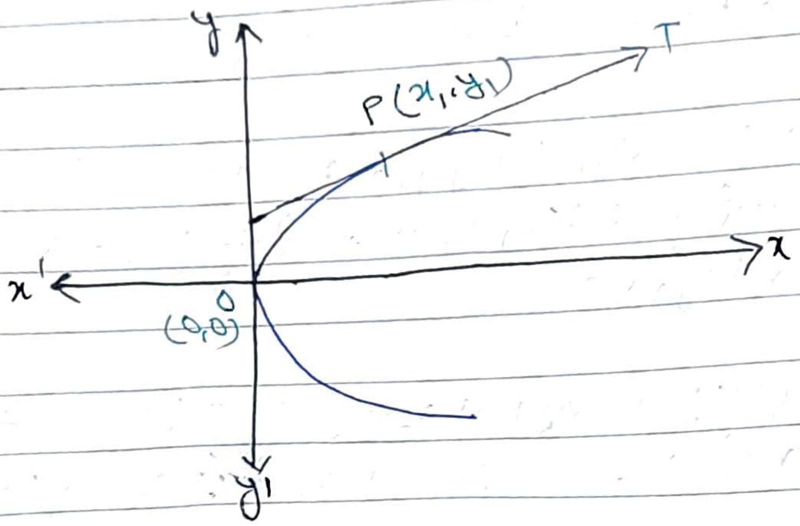
$\therefore \boxed{axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c =}$

[$\because$ The pt. (x_1, y_1) lies on the conic so it satisfy its equation
i.e. $ax_1^2 + by_1^2 + 2hx_1y_1 + 2gx_1 + 2fy_1 + c = 0$]

PARTICULAR CASES.

A) To find the eq^n of tangent to the parabola $y^2 = 4ax$ at any point (x_1, y_1)

Proof:



given equation of parabola is

$$S \equiv y^2 = 4ax$$

$$S \equiv y^2 - 4ax = 0 \quad \text{——— (1).}$$

diff (1) partially w.r.t x & y we get

$$\frac{\partial S}{\partial x} = -4a$$

$$\frac{\partial S}{\partial y} = 2y$$

Let PT be the tangent to the parabola at any point $P(x_1, y_1)$ then its eq^n is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

$$= -\left(\frac{\frac{\partial S}{\partial x}}{\frac{\partial S}{\partial y}}\right)_{(x_1, y_1)} (x - x_1)$$

$= -1\left(-\frac{4a}{2y_1}\right)(x-x_1)$

$= \frac{2a}{y_1}(x-x_1)$

$(y-y_1)y_1 = 2a(x-x_1)$

or, $yy_1 - y_1^2 = 2ax - 2ax_1$

or, $yy_1 = 2ax + y_1^2 - 2ax_1$

or, $yy_1 = \cancel{2ax} + y_1^2 - 2ax_1 + \cancel{2ax_1} - 2ax_1$

or, $yy_1 = 2ax + \cancel{y_1^2} + 4ax_1 \cancel{-} 2ax_1$

[$\because$ The pt. $P(x_1, y_1)$ lies on the parabola so, it satisfy its eq^n i.e. $y_1^2 = 4ax_1$).

$\therefore yy_1 = 2ax + 2ax_1$

$\boxed{yy_1 = 2a(x+x_1)}$

Corollary.

i) The eq^n of tangent to the parabola $y^2 = 4ax$ at any point $(at^2, 2at)$ is

$yy_1 = 2a(x+x_1)$

$y \cdot 2at = 2a(x + at^2)$

$\boxed{yt = x + at^2}$

2) The eq^n of tangent to the parabola $y^2 = 4ax$ at any pt. $(a/m^2, 2a/m)$ is

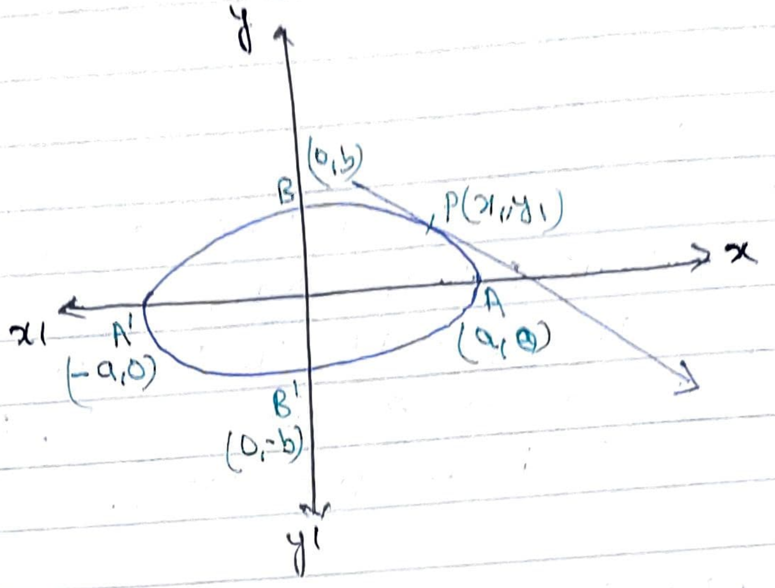
$y \cdot \frac{2a}{m} = 2a\left(x + \frac{a}{m^2}\right)$

$\frac{y}{m} = x + \frac{a}{m^2}$

$\boxed{my = m^2x + a}$

B) To find the equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point (x_1, y_1)

Proof:



given equation of ellipse is

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \quad \text{—— (1)}$$

diff (1) partially w.r.t. x and y we get

$$\frac{\partial S}{\partial x} = \frac{2x}{a^2}$$

$$\frac{\partial S}{\partial y} = \frac{2y}{b^2}$$

let PT be the tangent to the ~~parabola~~ ellipse at any point $P(x_1, y_1)$ then its eqⁿ is

$$y-y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$$

(14)

$$y - y_1 = \left(- \frac{\frac{\partial S}{\partial x}}{\frac{\partial S}{\partial y}} \right)_{(x_1, y_1)} (x - x_1)$$

$$= - \left\{ \frac{2x/a^2}{2y/b^2} \right\}_{(x_1, y_1)} (x - x_1)$$

$$= \frac{-b^2}{a^2} \frac{x_1}{y_1} (x - x_1)$$

or, $$\frac{(y - y_1) y_1}{b^2} = - \frac{x_1 (x - x_1)}{a^2}$$

or, $$\frac{y y_1 - y_1^2}{b^2} = \frac{- x x_1 + x_1^2}{a^2}$$

or, $$\frac{y y_1}{b^2} - \frac{y_1^2}{b^2} = - \frac{x x_1}{a^2} + \frac{x_1^2}{a^2}$$

or, $$\frac{y y_1}{b^2} + \frac{x x_1}{a^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

$\therefore$ $$\boxed{\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1}$$

[$\because$ the pt $P(x_1, y_1)$ lies on the ellipse, so it satisfy its eq^n i.e. $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$.

Corollary.

The eq^n of tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at any pt $(a\cos\phi, b\sin\phi)$ is

$$\frac{x . a\cos\phi}{a^2} + \frac{y\, b\sin\phi}{b^2} = 1$$

$\therefore$ $$\boxed{\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1}$$